

# Warped product contact $CR$ -submanifolds of globally framed $f$ -manifolds with Lorentz metric

Khushwant Singh and S. S. Bhatia

*School of Mathematics and Computer Applications*

*Thapar University, Patiala - 147 004, India*

E-mail: khushwantchahil@gmail.com, ssbhatia@thapa.redu

## Abstract

In the present paper, we study globally framed  $f$ -manifolds in the particular setting of indefinite  $S$ -manifolds for both spacelike and timelike cases. We prove that if  $M = N^\perp \times_f N^T$  is a warped  $CR$ -submanifold such that  $N^\perp$  is  $\phi$ -anti-invariant and  $N^T$  is  $\phi$ -invariant, then  $M$  is a  $CR$ -product. We show that the second fundamental form of a contact  $CR$  warped product of a indefinite  $S$  space form satisfies a geometric inequality,  $\|h\|^2 \geq p\{3\|\nabla \ln f\|^2 - \Delta \ln f + (c+2)k+1\}$ .

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## 1 Introduction

It is well-known that the notion of warped products plays some important role in differential geometry as well as physics. R.L. Bishop and B. O'Neill in 1969 introduced the concept of a warped product manifold to provide a class of complete Riemannian manifolds with everywhere negative curvature. The warped product scheme was later applied to semi-Riemannian geometry ([1],[2]) and general relativity [3]. Recently, Chen [10] (see also [11]) studied warped product  $CR$ -submanifolds and showed that there exist no warped product  $CR$ -submanifolds of the form  $M = N^\perp \times_f N^T$  such that  $N^\perp$  is a totally real submanifold and  $N^T$  is a holomorphic submanifold of a Kaehler manifold  $\tilde{M}$ . Therefore he considered warped product  $CR$ -submanifold in the form  $M = N^T \times_f N^\perp$  which is called  $CR$ -warped product, where  $N^T$  and  $N^\perp$  are holomorphic and totally real submanifolds of a Kaehler manifold  $\tilde{M}$ . Motivated by Chen's papers many authors studied  $CR$ -warped product submanifolds in almost complex as well as contact setting (see [8], [13]).

## 2 Globally framed $f$ -manifolds with Lorentz metric

First we recall some definitions due ([1],[6],[7],[9]).

A  $(2n + 1)$ -dimensional  $C^\infty$  manifold  $\bar{M}$  is said to have an *almost contact structure* if there exist on  $\bar{M}$  a tensor field  $\phi$  of type  $(1, 1)$ , a vector field  $\xi$  and a 1-form  $\eta$  satisfying

$$\phi^2 = -I + \eta \otimes \xi, \quad \phi\xi = 0, \quad \eta \circ \phi = 0, \quad \eta(\xi) = 1.$$

There always exists a Riemannian metric  $g$  on an almost contact manifold  $\bar{M}$  satisfying the following compatibility condition

$$\eta(X) = g(X, \xi), \quad g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y)$$

where  $X$  and  $Y$  are vector fields on  $\bar{M}$ .

Moreover, if  $g$  is a semi-riemannian metric on  $\bar{M}^{2n+1}$  such that,

$$g(\phi X, \phi Y) = g(X, Y) - \epsilon \eta(X)\eta(Y),$$

where  $\epsilon = \pm 1$  according to the causal character of  $\xi$ ,  $\bar{M}^{2n+1}$  is called an indefinite almost contact manifold if  $d\eta = \Phi$ ,  $\Phi$  being defined by  $\Phi(X, Y) = g(X, fY)$ .

In the Riemannian case a generalization of these structures have been studied by Blair [6], by Goldberg and Yano [12]. Brunetti and Pastore [9] studied such structures in semi-Riemannian context.

A manifold  $\tilde{M}$  is called a globally framed  $f$ -manifold (briefly  $g.f.f$ -manifold) if it is endowed with a non null  $(1, 1)$ -tensor field  $\phi$  of constant rank, such that  $\ker \phi$  is parallelizable i.e. there exist global vector field  $\xi_\alpha$ , such that  $\alpha = \{1, \dots, s\}$ , and 1-form  $\eta^\alpha$ , satisfying

$$(2.1) \quad \phi^2 = -I + \eta^\alpha \otimes \xi_\alpha \quad \text{and} \quad \eta^\alpha(\xi_\beta) = \delta^\alpha_\beta.$$

A  $g.f.f$ -manifold  $(\tilde{M}^{2n+s}, \phi, \eta^\alpha, \xi_\alpha)$ , such that  $\alpha \in \{1, \dots, s\}$ , is said to be an indefinite  $g.f.f$ -manifold if  $g$  is a semi-Riemannian metric satisfying the following compatibility condition

$$(2.2) \quad g(\phi X, \phi Y) = g(X, Y) - \epsilon_\alpha \eta^\alpha(X)\eta^\alpha(Y),$$

for any vector fields  $X, Y$ , where  $\epsilon_\alpha = \pm 1$  according to whether  $\xi_\alpha$  is spacelike or timelike. Then for any  $\alpha \in \{1, \dots, s\}$ ,

$$\eta^\alpha(X) = \epsilon_\alpha g(X, \xi_\alpha).$$

**Note:** We will consider  $\alpha \in \{1, \dots, s\}$  throughout the paper.

An indefinite  $g.f.f$ -manifold is an indefinite  $S$ -manifold if it is normal and  $d\eta^\alpha = \Phi$ , for any  $\alpha \in \{1, \dots, s\}$ , where  $\Phi(X, Y) = g(X, \phi Y)$  for any  $X, Y \in \chi(M^{2n+s})$ . The Levi-Civita connection of an indefinite  $S$ -manifold satisfies

$$(2.3) \quad (\nabla_X \phi)Y = g(\phi X, \phi Y)\bar{\xi}_\alpha + \bar{\eta}^\alpha(Y)\phi^2(X),$$

where  $\bar{\xi} = \sum_{\alpha=1}^s \xi_\alpha$  and  $\bar{\eta} = \epsilon_\alpha \eta^\alpha$ . Note that for  $s = 1$ , indefinite  $S$ -manifold becomes indefinite Sasaki manifold.

From (2.3), it follows that  $\nabla_X \xi_\alpha = -\epsilon_\alpha \phi X$  and  $\ker \phi$  is an integrable flat distribution since  $\nabla_{\xi_\alpha} \xi_\beta = 0$ , for any  $\alpha, \beta \in \{1, \dots, s\}$ . A  $g.f.f$ -manifold is subject to the topological condition: It has to be either non compact or compact with vanishing Euler characteristic, since it admits never vanishing vector fields. This implies that such a  $g.f.f$ -manifold always admit Lorentz metrics.

An indefinite  $S$ -manifold  $(\tilde{M}, \phi, \xi_\alpha, \eta^\alpha, g)$  is said to be an indefinite  $S$ -space if the  $\phi$ -sectional curvature  $H_p(X)$  is constant, for any point and any  $\phi$ -plane. In particular, in [9] it is proved that an indefinite  $S$ -manifold  $(M, \phi, \xi_\alpha, \eta^\alpha, g)$  is an indefinite  $S$ -space form with  $H_p(X) = c$  if and only if the Riemannian  $(0, 4)$ -type curvature tensor field  $R$  is given by

$$(2.4) \quad \begin{aligned} R(X, Y, Z, W) = & -\frac{c+3\epsilon}{4}\{g(\phi Y, \phi Z)g(\phi X, \phi W) - g(\phi X, \phi Z)g(\phi Y, \phi W)\} \\ & - \frac{c-\epsilon}{4}\{\Phi(W, X)\Phi(Z, Y) - \Phi(Z, X)\Phi(W, Y) \\ & + 2\Phi(X, Y)\Phi(W, Z)\} - \{\tilde{\eta}(W)\tilde{\eta}(X)g(\phi Z, \phi Y) \\ & - \tilde{\eta}(W)\tilde{\eta}(Y)g(\phi Z, \phi X) + \tilde{\eta}(Y)\tilde{\eta}(Z)g(\phi W, \phi X) \\ & - \tilde{\eta}(Z)\tilde{\eta}(X)g(\phi W, \phi Y)\}, \end{aligned}$$

for any vector fields  $X, Y, Z$  and  $W$  on  $M$ , where  $\epsilon = \sum_{\alpha=1}^s \epsilon_\alpha$ .

Let  $M$  be a real  $m$ -dimensional submanifold of  $\tilde{M}^{2n+s}$ , tangent to the global vector field  $\xi_\alpha$ . We shall need the Gauss and Weigarten formulae

$$(2.5) \quad \tilde{\nabla}_X Y = \nabla_X Y + h(X, Y), \quad \tilde{\nabla}_X N = -A_N X + \nabla_X^\perp N,$$

for any  $X, Y \in \chi(M)$  and  $N \in \Gamma^\infty(T(M)^\perp)$ , where  $\nabla^\perp$  is the connection on the normal bundle  $T(M)^\perp$ ,  $h$  is the second fundamental form and  $A_N$  is the Weingarten map associated with the vector field  $N \in T(M)^\perp$  as

$$g(A_N X, Y) = \tilde{g}(h(X, Y), N).$$

For any  $X \in \chi(M)$  we set  $PX = \tan(\phi X)$  and  $FX = \text{nor}(\phi X)$ , where  $\tan_x$  and  $\text{nor}_x$  are the natural projections associated to the direct sum decomposition

$$T_x(\tilde{M}) = T_x(M) \oplus T(M)_x^\perp, \quad x \in M.$$

Then  $P$  is an endomorphism of the tangent bundle of  $T(M)$  and  $F$  is a normal bundle valued 1-form on  $M$ . Since  $\xi_\alpha$  is tangent to  $M$ , we get

$$P\xi_\alpha = 0, \quad F\xi_\alpha = 0, \quad \nabla_X \xi_\alpha = PX, \quad h(X, \xi_\alpha) = FX.$$

Similarly, for a normal vector field  $F$ , we put  $tF = \tan(\phi F)$  and  $fF = \text{nor}(\phi F)$  for the tangential and normal part of  $\phi F$ , respectively.

The covariant derivative of the morphisms  $P$  and  $F$  are defined respectively as

$$(\nabla_U P)V = \nabla_U PV - P\nabla_U V, \quad (\nabla_U F)V = \nabla_U^\perp FV - F\nabla_U V,$$

for  $U, V \in \chi(M)$ . On using equation (2.3) and (2.5) we get

$$(2.6) \quad (\nabla_U P)V = g(PU, PV)\xi_\alpha + \eta^\alpha(U)\eta^\alpha(V)\xi_\alpha - \eta^\alpha(V)U - th(U, V) - A_{FV}U$$

and

$$(2.7) \quad (\nabla_U F)V = g(FU, FV)\xi_\alpha - fh(U, V) - h(U, PV).$$

The Riemannian curvature tensor  $R$  of  $M$  is given by

$$(2.8) \quad R_{XY}Z = \frac{3\epsilon^2 + c\epsilon - 4}{4}\{g(Y, Z)\eta^\alpha(X)\xi_\alpha - g(X, Z)\eta^\alpha(Y)\xi_\alpha \\ + \eta^\alpha(Y)\eta^\alpha(Z)X - \eta^\alpha(X)\eta^\alpha(Z)Y + \frac{c+3\epsilon}{4}\{g(X, Z)Y - g(Y, Z)X\} \\ - \frac{c-\epsilon}{4}\{g(Z, PY)PX - g(Z, PX)PY - 2g(X, PY)PZ\},$$

for all  $X, Y, Z$  vector fields on  $M$ , we recall the equation of Gauss and Codazzi, respectively

$$(2.9) \quad \tilde{g}(\tilde{R}_{XY}Z, W) = g(R_{XY}Z, W) - g(h(X, W), h(Y, Z)) + g(h(Y, W), h(X, Z))$$

$$(2.10) \quad (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) = (\tilde{R}_{XY}Z)^\perp,$$

where  $(\nabla)h$  the covariant derivative of the second fundamental form is given by

$$(2.11) \quad ((\tilde{\nabla}_X h)(Y, Z) = \bar{\nabla}_X h(Y, Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z),$$

for all  $X, Y, Z \in TM$ . Codazzi equation becomes

$$(2.12) \quad (\tilde{\nabla}_X h)(Y, Z) - (\tilde{\nabla}_Y h)(X, Z) = \frac{c - \epsilon}{4} \{g(PX, Z)FY - g(PY, Z)FX + 2g(X, PY)FZ\}.$$

The second fundamental form  $h$  satisfies the classical Codazzi equation (according to [3]) if

$$(2.13) \quad (\tilde{\nabla}_X h)(Y, Z) = (\tilde{\nabla}_Y h)(X, Z).$$

**Lemma 2.1** *Let  $M^m$  be a submanifold of an indefinite  $S$ -space form  $\tilde{M}^{2m+s}(c)$  tangent to the global vector field  $\xi_\alpha$  with  $c \neq 1, -1$  according to whether  $\xi_\alpha$  is spacelike or timelike. If the second fundamental form  $h$  of  $M^m$  satisfies the classical Codazzi equation then  $M^m$  is  $\phi$ -invariant or  $\phi$ -anti-invariant.*

**Proof.** By using (2.12) and (2.13), we get

$$(2.14) \quad \{g(PX, Z)FY - g(PY, Z)FX + 2g(X, PY)FZ\} = 0,$$

for all  $X, Y, Z \in T(M)$ . By contradiction, let there exist  $U_x \in T_x(M)$  such that  $PU_x \neq 0$  and  $FU_x \neq 0$ . From (2.14), we deduce  $2g(U_x, PU_x)FU_x = 0$ , which is false. Therefore, for  $U_x \in T_x(M)$ , we have either  $PU_x = 0$  or  $FU_x = 0$ . It can be also proved that we can not have  $U_x, V_x \in T_x(M)$  such that  $PU_x \neq 0$ ,  $FU_x = 0$ ,  $PV_x = 0$  and  $FV_x \neq 0$ . Therefore either  $P = 0$  or  $F = 0$  which completes the statement.

**Lemma 2.2** *Let  $M^m$  be a contact  $CR$ -submanifold of an indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$ . Then for any  $Z, W \in D^\perp$ , we have*

$$(2.15) \quad A_{FZ}W + A_{FW}Z = \eta^\alpha(Z)W + \eta^\alpha(W)Z - 2\eta^\alpha(W)\eta^\alpha(Z)\xi_\alpha.$$

**Proof.** Proof is straightforward and can be obtained by using equation (1.3) and (1.5).

Clearly, for  $\xi_\alpha \in D$  we also have

$$(2.16) \quad A_{FZ}W + A_{FW}Z = 0,$$

where  $W, Z \in D^\perp$ .

**Lemma 2.3** *Let  $M^m$  be a contact  $CR$ -submanifold of an indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$  with  $\xi_\alpha \in D$ . Then the following are equivalent*

- (i)  $h(X, PY) = h(PX, Y) \quad \forall X, Y \in D$ ,
- (ii)  $\tilde{g}(h(X, PY), \phi Z) = \tilde{g}(h(PX, Y), \phi Z) \quad \forall X, Y \in D, \quad \forall Z \in D^\perp$ ,

(iii)  $D$  is completely Integrable.

**Proof.** Easy Calculations.

For a leaf of anti-invariant distribution  $D^\perp$ . We prove the following

**Proposition 2.4** *Let  $M^m$  be a contact CR-submanifold of an indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$ . Then any leaf of  $D^\perp$  is totally geodesic in  $M^m$  if and only if*

$$(2.17) \quad g(h(D, D^\perp), \phi D^\perp) = 0.$$

**Proof.** By hypothesis

$$(2.18) \quad g(P\nabla_W Z, Y) = -g(A_{FZ}W, Y) - g(th(Z, W), Y) = -g(h(Y, W), FZ),$$

for any  $Y \in D$ ,  $Z, W \in D^\perp$ . Then

$$(2.19) \quad g(\nabla_W Z, PY) = -g(h(Y, W), FZ).$$

Consequently,  $\nabla_W Z \in D^\perp$  if and only if (2.17) holds.

Let  $\nu$  be the complementary orthogonal subbundle of  $\phi D^\perp$  in the normal bundle  $T(M)^\perp$ . Thus we have the following direct sum decomposition

$$(2.20) \quad T(M)^\perp = \phi D^\perp \oplus \nu.$$

Similarly, we can also prove that,  $h(X, Y) \in \nu$  and  $\phi h(X, Y) = h(X, PY)$  for all  $X, Y$  tangent to  $N^T$ . On  $N^T$  we have an induced indefinite  $S$ -structure.

**Lemma 2.5** *Let  $M^m$  be a contact CR-submanifold of an indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$  with  $\xi_\alpha \in D$ . Then for all  $X, Y \in D$ , we have  $\phi h(X, Y) \in \phi D^\perp \oplus \nu$ .*

**Proof.** From lemma 2.5 it follows that  $\phi\nu = \nu$ . Since  $h$  is normal to  $M^m$  and  $\eta^\alpha(D^\perp) = 0$ . We easily get the result.

A contact CR-submanifold  $M^m$  of an indefinite  $S$ -manifold is called contact CR-product if it is locally a Riemannian product of a  $\phi$ -invariant submanifold  $N^T$  tangent to  $\xi_\alpha$  and a totally real submanifold  $N^\perp$  of  $\tilde{M}^{2n+s}$ .

**Theorem 2.6** *Let  $M^m$  be a contact CR-submanifold of an indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$  and set  $\xi_\alpha \in D$ . Then  $M^m$  is contact CR-product if and only if  $P$  satisfies*

$$(2.21) \quad (\nabla_U P)V = g(PU_D, PV)\xi_\alpha - \eta^\alpha(V)U_D + \eta^\alpha(U_D)\eta^\alpha(V)\xi_\alpha,$$

where  $U, V$  tangent to  $M^m$  and we are taking  $U_D$  as the component of  $D$ .

**Proof.** Since  $\phi \equiv P$  on  $N^T$ , due to indefinite  $S$ -structure of  $\tilde{M}^{2n+s}$  using the Gauss formula we get

$$(\nabla_X P)Y = g(PX, PY)\xi_\alpha - \eta^\alpha(Y)X + \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha + h(X, PY) - \phi h(X, Y),$$

for any  $X, Y \in N^T$ . Taking the components in  $D$  one gets

$$(2.22) \quad (\nabla_X P)Y = g(PX, PY)\xi_\alpha - \eta^\alpha(Y)X + \eta^\alpha(X)\eta^\alpha(Y)\xi_\alpha.$$

Consider now  $Z \in N^\perp$  and  $Y \in N^T$ . Similarly, we can prove

$$(2.23) \quad (\nabla_Z P)Y = -\eta^\alpha(Y)Z,$$

as consequence

$$h(Z, PY) = \phi h(Z, Y) + \eta(Y)Z, \quad \forall Y \in N^T, \quad Z \in N^\perp.$$

Now it is easy to show that  $(\nabla_U P)Z = 0$  for all  $U \in \chi(M)$ ,  $Z \in D^\perp$  and hence the conclusion.

Conversely, consider (2.21) exists. Let  $U = X$ ,  $V = Z$  with  $X \in D$  and  $Z \in D^\perp$ . The relation (2.21) becomes  $(\nabla_X P)Z = 0$  and by using (2.6) we obtain  $th(X, Z) = -A_{FZ}X$ . Considering  $U = Z$ ,  $V = X$  (with  $X, Z$  as above) we obtain  $(\nabla_Z P)X = -\eta^\alpha(X)Z$ . Thus one gets

$$(2.24) \quad A_{FZ}X = \eta^\alpha(X)Z,$$

for all  $X \in D$  and  $Z \in D^\perp$ . After the computations we obtain  $\tilde{g}(h(X, PY) - h(PX, Y)) = 0$ . Let  $X \in H(M)$ ,  $Z, W \in D^\perp$ . Due to (2.24) we have

$$\tilde{g}(h(X, Z), \phi W) = \tilde{g}(A_{FW}X, Z) = g(\eta^\alpha(X)W, Z) = \eta^\alpha(X)g(W, Z) = 0.$$

Thus by virtue of Proposition (2.1),  $N^\perp$  is totally geodesic in  $M^m$ . Let now  $X, Y \in D$ , from (2.21) and (2.6) we obtain  $th(X, Y) = 0$ . If  $Z \in D^\perp$  we have

$$0 = \tilde{g}(th(X, Y), Z) = \tilde{g}(\tilde{\nabla}_X Y, \phi Z) = -\tilde{g}(Y, (\tilde{\nabla}_X \phi)Z) - \tilde{g}(Y, \phi \tilde{\nabla}_X Z) = -\tilde{g}(\phi Y, \nabla_X Z),$$

replacing  $Y$  by  $\phi Y$  one obtains  $\tilde{g}(Y, \nabla_X Z) = 0$  for all  $X, Y \in D$  and  $Z \in D^\perp$ . It follows that  $g(\nabla_X Y, Z) = 0$  which means that  $N^T$  is also totally geodesic in  $M^m$ . We may conclude that  $M^m$  is a contact  $CR$ -product in  $\tilde{M}^{2n+s}$ .

**Proposition 2.7** *Let  $M^m$  be a contact  $CR$ -submanifold of an indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$  with  $\xi_\alpha \in D$ . Then  $M^m$  is a contact  $CR$ -product if and only if*

$$(2.25) \quad A_{\phi Z}X = \eta^\alpha(X)Z,$$

for all  $X \in D$  and  $Z \in D^\perp$ .

**Proof.** Suppose that (2.25) holds. We have

$$\tilde{g}(B(X, Z), \phi W) = \tilde{g}(A_{\phi W} X, Z) = \eta^\alpha(X)g(Z, W) = 0, \quad \forall X \in H(M), \quad \forall Z, W \in D^\perp.$$

From Proposition 2.1 we get that  $N^\perp$  (the integral manifold of  $D^\perp$ ) is totally geodesic in  $M^m$ . Consider now  $X, Y \in D$  and  $Z \in D^\perp$ . We have

$$\tilde{g}(B(X, \phi Y), \phi Z) = \tilde{g}(A_{\phi Z} X, \phi Y) = \tilde{g}(\eta^\alpha(X)Z, \phi Y) = 0.$$

Similarly  $\tilde{g}(B(Y, \phi X), \phi Z) = 0$  and by Lemma 2.3 it follows that  $D$  is completely integrable. To prove that  $N^T$  (the integral manifold of  $D$ ) is totally geodesic in  $M^m$  we will prove that  $\nabla_X Y$  belongs to  $N^T$  for all  $X, Y$  tangent to  $N^T$ . We have  $g(\nabla_X Y, Z) = -g(Y, \nabla_X Z)$ . On the other hand, from the hypothesis  $\tilde{g}(B(X, Y), \phi Z) = 0$ . Then

$$\tilde{g}(B(X, Y), \phi Z) = -\tilde{g}(Y, \tilde{\nabla}_X(\phi Z)) = -\tilde{g}(\phi Y, \tilde{\nabla}_X Z) = -\tilde{g}(\phi Y, \nabla_X Z).$$

So, we obtain  $g(\phi Y, \nabla_X Z) = 0$ , for all  $X, Y \in D$  and for all  $Z \in D^\perp$ . But  $g(\xi_\alpha, \nabla_X Z) = 0$  and hence  $g(Y, \nabla_X Z) = 0$ . We may conclude now that  $\nabla_X Y \in N^T$  for all  $X, Y \in N^T$ . Therefore the two integral manifolds  $N^T$  and  $N^\perp$  are both totally geodesic in  $M^m$ . Consequently,  $M^m$  is locally a Riemannian product of  $N^T$  and  $N^\perp$ .

Conversely, from the totally geodesy of  $N^T$  and  $N^\perp$ , using the Gauss formula we get  $\tilde{g}(\tilde{\nabla}_X Y, \phi Z) = \tilde{g}(B(X, Y), \phi Z)$  with  $X, Y \in D$  and  $Z \in D^\perp$ . The right side is exactly  $g(A_{\phi Z} X, Y)$  while the left side equals to  $-\tilde{g}(\phi Y, \tilde{\nabla}_X Z) = g(\nabla_X(\phi Y), Z) = 0$ .

It follows that  $A_{\phi Z} X \in D^\perp$ . Again by using the Gauss formula we obtain after the computations  $\eta^\alpha(X)g(Z, W) = \tilde{g}(A_{\phi Z} X, W)$ . Taking into account that  $A_{\phi Z} X \in D^\perp$  it follows  $A_{\phi Z} X = \eta^\alpha(X)Z$ . This completes the proof.

Now we will prove the geometrical description of contact  $CR$ -products in indefinite  $S$ -space forms.

**Theorem 2.8** *Let  $M^m$  be a generic, simply connected contact  $CR$ -submanifold of an indefinite  $S$ -space form  $\tilde{M}^{2n+s}(c)$ , If  $M^m$  is a contact  $CR$ -product then*

(i) *For spacelike global vector field  $\xi_\alpha$*

$$\left(\frac{c+3}{4}\right)(g(PX, PZ)FY - g(PY, PZ)FX) = 0.$$

(ii) *For timelike global vector field  $\xi_\alpha$*

$$\left(\frac{c+5}{4}\right)(g(PX, PZ)FY - g(PY, PZ)FX) = 0.$$



Therefore if  $c \neq -3, -5$  and  $M^m$  is a  $\phi$ -anti-invariant submanifold of  $\tilde{M}^{2n+s}$ , then  $M^m$  is locally a Riemannian product of an integral curve of  $\xi_\alpha$  and a totally real submanifold  $N^\perp$  of  $\tilde{M}^{2n+s}$  and if  $c = -3, -5$ , then  $M^m$  is locally a Riemannian product of  $N^T$  and  $N^\perp$ .

**Proof.** Since  $M^m$  is generic it follows that  $\xi_\alpha \in D$ . By remark 2.1 we have  $h(X, Y) = 0$  for all  $X, Y \in D$  and  $A_{FZ}X = \eta^\alpha(X)Z$  for all  $X \in D$  and  $Z \in D^\perp$ . Since  $T(M)^\perp = \phi D^\perp$  and  $h \in T(M)^\perp$  by using the Weingarten formula we immediately see that  $g(h(X, Z), \phi W) = g(A_{\phi W}X, Z) = \eta^\alpha(X)g(W, Z)$ .

Consequently  $h(X, Z) = \eta^\alpha(X)\phi Z$  for all  $X \in D$  and  $Z \in D^\perp$ .

By making use of (2.11) we obtain for  $X, U, V \in T(M)$ .

$$\begin{aligned} (2.26) \quad (\nabla_X h)(U, PV) &= -h(U, (\nabla_X P)V + P\nabla_X V) \\ &= g(U, g(PX, PV)\xi_\alpha - \eta^\alpha(V)X + \eta^\alpha(X)\eta^\alpha(V)\xi_\alpha) \\ &= g(PX, PV)\eta^\alpha(U) - \eta^\alpha(V)g(U, X) + \eta^\alpha(X)\eta^\alpha(V)\eta^\alpha(U), \end{aligned}$$

hence we get

$$(2.27) \quad (\nabla_X h)(U, PV) = g(PX, PV)FU - \eta^\alpha(V)g(U, X) + \eta^\alpha(X)\eta^\alpha(V)\eta^\alpha(U)$$

Substitute in (2.12)  $Z$  by  $PZ$  (with  $Z \in T(M)$  arbitrary) the following identity holds:

$$(\nabla_X h)(Y, PZ) - (\nabla_Y h)(X, PZ) = \frac{c-\epsilon}{4} \{g(PX, PZ)FY - g(PY, PZ)FX\},$$

where  $\epsilon = \pm 1$  according to whether  $\xi_\alpha$  is spacelike or timelike. Combining with (2.27) above relation yields to

$$g(PX, PZ)FY - g(PY, PZ)FX = \frac{c-\epsilon}{4} \{g(PX, PZ)FY - g(PY, PZ)FX\},$$

which is equivalent to

$$(2.28) \quad \left(\frac{c-\epsilon}{4} + 1\right)(g(PX, PZ)FY - g(PY, PZ)FX) = 0.$$

Now we have to discuss two situations:  $\epsilon = \pm 1$ . For spacelike global vector field  $\epsilon = +1$ , above equation becomes

$$(2.29) \quad \left(\frac{c+3}{4}\right)(g(PX, PZ)FY - g(PY, PZ)FX) = 0.$$

For timelike global vector field  $\epsilon = -1$ , the above equation becomes

$$(2.30) \quad \left(\frac{c+5}{4}\right)(g(PX, PZ)FY - g(PY, PZ)FX) = 0.$$

Now we discuss the two cases

**Case I.** For  $c \neq -3, -5$ , From the equation (2.29) and (2.30) we obtain  $g(PY, PZ)FX - g(PX, PZ)FY = 0$ , for all  $X, Y, Z \in T(M)$ : Since  $M^m$  is generic we have  $F \neq 0$  and it is not difficult to prove that  $P = 0$ , thus  $M^m$  is  $\phi$ -anti-invariant. Moreover, by Theorem 2.6 we can say that  $M^m$  is a contact  $CR$ -product of an integral curve of  $\xi_\alpha$  and a totally real submanifold  $N^\perp$  of  $\tilde{M}^{2n+s}$ .

**Case II.** For  $c = -3, -5$  from (2.29) and (2.30).  $M$  is a contact  $CR$ -product of the invariant submanifold  $N^T$  and the anti-invariant submanifold  $N^\perp$ : Since  $N^T$  is totally geodesic in  $M^m$  and  $h(X, Y) = 0$  for all  $X, Y \in D$  then  $N^T$  is totally geodesic in  $\tilde{M}^{2n+s}$ . Thus, we can use the well known result that  $M^m$  has constant  $\phi$ -sectional curvature, then  $M^m$  is simply connected and hence  $M^m$  is the Riemannian product of  $N^T$  and  $N^\perp$ .

Let  $\tilde{H}_h(U, V)$  be the  $\phi$ -holomorphic bisectional curvature of the plane  $U \wedge V$ , i.e.

$$\tilde{H}_h(U, V) = \tilde{R}(\phi U, U, \phi V, V) \quad \text{for } U, V \in T(M).$$

We prove the following important lemmas for later use.

**Lemma 2.9** *Let  $M^m$  be a contact  $CR$ -product of a indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$ . Then, for any unit vector fields  $X \in D$  and  $Z \in D^\perp$ , then:*

$$(2.31) \quad \tilde{g}(h(\nabla_{\phi X} X, \phi Z), Z) = -s, \quad \tilde{g}(h(\nabla_X \phi X, \phi Z), Z) = s,$$

$$(2.32) \quad \tilde{g}(h(X, \nabla_{\phi X} \phi Z), Z) = 0, \quad \tilde{g}(h(\phi X, \nabla_X \phi Z), Z) = 0.$$

**Proof.** By theorem (2.6)

$$\begin{aligned} \tilde{g}(h(\nabla_{\phi X} X, \phi Z), Z) &= \sum_{\alpha=1}^s \eta^\alpha(\nabla_{\phi X} X) g(\phi^2 Z, Z) = - \sum_{\alpha=1}^s g(X, \nabla_{\phi X} \xi_\alpha) g(Z, Z) \\ &= - \sum_{\alpha=1}^s g(\phi X, \phi X) = -s, \end{aligned}$$

which is first part of (2.31). We also have

$$\begin{aligned} \tilde{g}(h(\nabla_X \phi X, \phi Z), Z) &= \sum_{\alpha=1}^s \eta^\alpha(\nabla_X \phi X) g(Z, Z) = - \sum_{\alpha=1}^s g(\phi X, \nabla_X \xi_\alpha) g(Z, Z) \\ &= \sum_{\alpha=1}^s g(\phi X, \phi X) = s, \end{aligned}$$

which is other part of (2.31). Finally

$$\begin{aligned}\tilde{g}(h(X, \nabla_{\phi X} \phi Z), Z) &= \sum_{\alpha=1}^s \eta^\alpha(X) g(\nabla_{\phi X} \phi Z, Z) = 0, \\ \tilde{g}(h(\phi X, \nabla_X \phi Z), Z) &= \sum_{\alpha=1}^s \eta^\alpha(\phi X) g(\nabla_X \phi Z, Z) = 0,\end{aligned}$$

which are (2.32).

**Lemma 2.10** *Let  $M^m$  be a contact CR-product of a indefinite S-manifold  $\tilde{M}^{2n+s}$ . Then, for any unit vector fields  $X \in D$  and  $Z \in D^\perp$  we have*

$$(2.33) \quad \tilde{H}_h(X, Z) = 2s - 2\|h(X, Z)\|^2$$

**Proof.** We know

$$\tilde{R}(\phi X, X, \phi Z, Z) = \tilde{g}((\nabla^h_{\phi X} h)(X, \phi Z) - (\nabla^h_X h)(\phi X, \phi Z), Z)$$

by using (2.11) and above lemma, we gets

$$\begin{aligned}\tilde{R}(\phi X, X, \phi Z, Z) &= \tilde{g}(\tilde{\nabla}_{\phi X} h(X, \phi Z) - h(\nabla_{\phi X} X, \phi Z) - h(X, \nabla_{\phi X} \phi Z), Z) \\ &\quad - \tilde{g}(\tilde{\nabla}_X h(\phi X, \phi Z) - h(\nabla_X \phi X, \phi Z) - h(\phi X, \nabla_X \phi Z), Z) \\ &= 2s - \tilde{g}(h(X, \phi Z), \nabla_{\phi X} Z) - \tilde{g}(h(X, \phi Z), h(\phi X, Z)) \\ &\quad + g(h(\phi X, \phi Z), \nabla_X Z) + g(h(\phi X, \phi Z), h(X, Z)).\end{aligned}$$

Now, using the fact that  $\nabla_X Z$  and  $\nabla_{\phi X} Z$  belong to  $D^\perp$ , therefore above equation becomes

$$\tilde{R}(\phi X, X, \phi Z, Z) = 2s - 2\|h(X, Z)\|^2,$$

this ends the proof.

We come to know that  $\tilde{H}_h(U, \xi_\alpha) = 0$  and  $h(U, \xi_\alpha) = \phi U$ , So, when we will refer to the  $\phi$ -holomorphic bisectonal curvature of the plane  $U \wedge V$ , we intend that this plane is orthogonal  $\xi_\alpha$ . Thus for  $X$  in the above lemma we can suppose that it belongs to  $H(M)$ .

**Proposition 2.11** *Let  $\tilde{M}^{2n+s}(c)$  be a indefinite S-space form and let  $X, Z$  be two unit vector fields orthogonal to global vector field  $\xi_\alpha$ . Then the  $\phi$ -holomorphic bisectonal curvature of the plane  $X \wedge Z$  is given by*

(i) *For spacelike global vector field  $\xi_\alpha$*

$$\tilde{H}_B(X, Z) = \frac{c-1}{4}g(\phi X, Z) - \frac{c+3}{4}g(\phi X, Z)^2 + c.$$

(ii) For timelike global vector field  $\xi_\alpha$

$$\tilde{H}_B(X, Z) = \frac{c+1}{4}g(\phi X, Z) - \frac{c-3}{4}g(\phi X, Z)^2 + c.$$

**Corollary 2.12** *Let  $\tilde{M}^{2n+s}(c)$  be a indefinite  $S$ -space form and let  $X \in H(M)$  and  $Z \in D^\perp$  be unit vector fields orthogonal to global vector field  $\xi_\alpha$ . Then the  $\phi$ -holomorphic bisectional curvature of the plane  $X \wedge Z$  for spacelike and timelike global vector field  $\xi_\alpha$  is given by*

$$\tilde{H}_B(X, Z) = c.$$

**Theorem 2.13** *Let  $\tilde{M}^{2m+s}(c)$  be a indefinite  $S$ -space form and let  $M = N^T \times N^\perp$  be a contact  $CR$ -product in  $\tilde{M}^{2n+s}$ . Then the norm of the second fundamental form of  $M$  satisfies the inequality*

$$\|h\|^2 \geq ((3c + 8s - 3\epsilon)p + 2s)q,$$

where  $\epsilon = \pm 1$  according to whether global vector field  $\xi_\alpha$  is spacelike or timelike. The equality sign holds if and only if both  $N^T$  and  $N^\perp$  are totally geodesic.

**Proof.** For  $X \in H(M)$  and  $Z \in D^\perp$  we have  $\|h(X, Z)\|^2 = \frac{1}{4}(3c + 8s - 3\epsilon)$ .

Now, we choose a local field of orthonormal frames

$$\begin{aligned} \{X_1, \dots, X_p, X_{p+1} = \phi X_1, \dots, X_{2p} = \phi X_p, X_{2p+1} = Z_1, \dots, X_n = Z_q, \\ X_{n+1} = \phi Z_1, \dots, X_{n+q} = \phi Z_q, X_{n+q+1}, \dots, X_{2m}, \xi_1, \dots, \xi_s\} \end{aligned}$$

on  $\tilde{M}^{2n+s}(c)$  in such a way that  $\{X_1, \dots, X_{2p}\}$  is a local frame field on  $D$  and  $\{Z_1, \dots, Z_q\}$  is a local frame field on  $D^\perp$ . Thus

$$\begin{aligned} \|h\|^2 &= \|h(D, D)\|^2 + \|h(D^\perp, D^\perp)\|^2 + 2\|h(D, D^\perp)\|^2 \geq 2\|h(D, D^\perp)\|^2 \\ &= 2\left(\sum_{i=1}^{2p} \sum_{j=1}^q \|h(X_i, Z_j)\|^2 + \sum_{\alpha=1}^s \sum_{j=1}^q \|h(\xi_\alpha, Z_j)\|^2\right) \\ &= ((3c + 8s - 3\epsilon)p + 2s)q, \end{aligned}$$

where  $X_i$  and  $Z_j$  are orthonormal basis in  $H(M)$  and  $D^\perp$  respectively. The equality sign holds if and only if  $h(D, D) = 0$  and  $h(D^\perp, D^\perp) = 0$ , which is equivalent to the totally geodesy of  $N^T$  and  $N^\perp$ .

### 3 $CR$ -warped product submanifolds in indefinite $S$ -manifolds

The main purpose of this section is devoted to the presentation of some properties of warped product contact  $CR$ -submanifolds in indefinite  $S$ -manifolds.

Let  $(N_1, g_1)$  and  $(N_2, g_2)$  be two Riemannian manifolds and  $f$ , a positive differentiable function on  $N_1$ . The *warped product* of  $N_1$  and  $N_2$  is the product manifold  $N_1 \times_f N_2 = (N_1 \times N_2, g)$ , where

$$(3.1) \quad g = g_1 + f^2 g_2,$$

where  $f$  is called the *warping function* of the warped product. The warped product  $N_1 \times_f N_2$  is said to be *trivial* or simply Riemannian product if the warping function  $f$  is constant. This means that the Riemannian product is a special case of warped product.

We recall the following general results obtained by Bishop and O'Neill [2] for warped product manifolds.

**Lemma 3.1** *Let  $M = N_1 \times_f N_2$  be a warped product manifold with the warping function  $f$ , then*

- (i)  $\nabla_X Y \in TN_1$  for each  $X, Y \in TN_1$ ,
- (ii)  $\nabla_X Z = \nabla_Z X = (X \ln f)Z$ , for each  $X \in TN_1$  and  $Z \in TN_2$ ,
- (iii)  $\nabla_Z W = \nabla_Z^{N_2} W - \frac{g(Z, W)}{f} \text{grad} f$ ,

where  $\nabla$  and  $\nabla^{N_2}$  denote the Levi-Civita connections on  $M$  and  $N_2$ , respectively.

In the above lemma  $\text{grad} f$  is the gradient of the function  $f$  defined by  $g(\text{grad} f, U) = Uf$ , for each  $U \in TM$ . From the Lemma 3.1, we have on a warped product manifold  $M = N_1 \times_f N_2$

- (i)  $N_1$  is totally geodesic in  $M$ .
- (ii)  $N_2$  is totally umbilical in  $M$ .

**Theorem 3.2** *Let  $\tilde{M}^{2n+s}$  be a indefinite  $S$ -manifold and let  $M = N^\perp \times_f N^T$  be a warped product  $CR$ -submanifold such that  $N^\perp$  is a totally real submanifold and  $N^T$  is  $\phi$ -holomorphic (invariant) of  $\tilde{M}^{2n+s}$ . Then  $M$  is a  $CR$ -product.*

**Proof.** Let  $X$  be tangent to  $N^T$  and let  $Z$  be a vector field tangent to  $N^\perp$ . From the above lemma we find that

$$(3.2) \quad \nabla_X Z = (Z \ln f)X.$$

Now we have two cases either  $\xi_\alpha$  is tangent to  $N^T$  or  $\xi_\alpha$  is tangent to  $N^\perp$ .

Case I.  $\xi_\alpha$  is tangent to  $N^T$ . Take  $X = \xi_\alpha$ . Since  $\nabla_Z \xi_\alpha = -PZ = 0$  and  $\nabla_Z \xi_\alpha = \nabla_{\xi_\alpha} Z$  ( $\xi_\alpha$  is tangent to  $N^T$  while  $Z$  is tangent to  $N^\perp$ ) one gets  $0 = Z(\ln f)\xi_\alpha$  and hence  $Z(\ln f) = 0$  for all  $Z$  tangent to  $N^\perp$ . Consequently  $f$  is constant and thus the warped product above is nothing but a Riemannian

product.

Case II. Now we will consider the other case,  $\xi_\alpha$  is tangent to  $N^\perp$ . Similarly, take  $Z = \xi_\alpha$ . Since  $\nabla_X \xi_\alpha = -\epsilon_\alpha PX = -\epsilon_\alpha \phi X$  it follows  $-\epsilon_\alpha \phi X = (\xi_\alpha \ln f)X$ . But this is impossible if  $\dim N^T \neq 0$ .

**Remark:** There do not exist warped product  $CR$ -submanifolds in the form  $N^\perp \times_f N^T$  other than  $CR$ -products such that  $N^T$  is a  $\phi$ -invariant submanifold and  $N^\perp$  is a totally real submanifold of  $\tilde{M}$ .

From now on we will consider warped product  $CR$ -submanifolds in the form  $N^T \times_f N^\perp$ .

We can say that *A contact  $CR$ -submanifold  $M$  of a indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$ , tangent to the structure vector field  $\xi_\alpha$  is called a contact  $CR$ -warped product if it is the warped product  $N^T \times_f N^\perp$  of an invariant submanifold  $N^T$ , tangent to  $\xi_\alpha$  and a totally real submanifold  $N^\perp$  of  $\tilde{M}^{2n+s}$ , where  $f$  is the warping function.*

**Lemma 3.3** *Let  $M^m$  be a contact  $CR$ -submanifold in indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$ , such that  $\xi_\alpha \in D$ . Then we have*

$$(3.3) \quad g(\nabla_U X, Z) = -\tilde{g}(\phi A_{\phi Z} U, X), \quad \forall X \in D, \quad \forall Z \in D^\perp, \quad \forall U \in T(M),$$

$$(3.4) \quad A_{\phi\mu} X + A_\mu \phi X = 0 \quad \forall X \in D, \quad \forall \mu \in \nu.$$

**Proof.** We have  $\tilde{g}(\phi A_{\phi Z} U, X) = \tilde{g}(A_{\phi Z} U, \phi X) = \tilde{g}(\nabla_U^\perp \phi Z - \tilde{\nabla}_U \phi Z, \phi X) = -\tilde{g}(\phi \tilde{\nabla}_U Z, \phi X) = -\tilde{g}(\tilde{\nabla}_U Z, X) + \eta^\alpha(\tilde{\nabla}_U Z) \eta^\alpha(X) = -g(\nabla_U Z, X) - \tilde{g}(Z, \tilde{\nabla}_U \xi) \eta^\alpha(X) = -g(\nabla_U Z, X) + g(Z, \phi U) \eta^\alpha(X) = -g(\nabla_U Z, X)$ . So, equation (3.3) follows.

For the proof of the equation (3.4) we have  $g(A_{\phi\mu} X, U) = -g(\tilde{\nabla}_X \phi \mu, U) = g(\mu, \phi \tilde{\nabla}_X U)$  and  $g(A_\mu U, \phi X) = -g(\mu, \phi \tilde{\nabla}_U X)$  with  $U \in T(M)$ . It follow that  $A_{\phi\mu} X + A_\mu \phi X = 0, \forall X \in D, \forall \mu \in \nu$ .

**Lemma 3.4** *If  $M = N^T \times_f N^\perp$  is a contact  $CR$ -warped product in a indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$  the for  $X$  tangent to  $N^T$  and  $Z, W$  tangent to  $N^\perp$  we have*

$$(3.5) \quad g(h(D, D^\perp), \phi D^\perp) = 0$$

$$(3.6) \quad \xi_\alpha(f) = 0$$

$$(3.7) \quad g(h(\phi X, Z), \phi W) = (X \ln f)g(Z, W)$$

**Proof.** For any  $X, Y \in D$  and  $Z \in D^\perp$ , we have

$$g(h(X, Y), \phi Z) = g(\tilde{\nabla}_X Y, \phi Z) = -g(\phi Y, \tilde{\nabla}_X Z) = g(\tilde{\nabla}_X \phi Y, Z) = 0.$$

We know that  $\nabla_U \xi_\alpha = -\epsilon_\alpha P U$ . It follows that  $\nabla_Z \xi_\alpha = 0$  for all  $Z$  tangent to  $N^\perp$ . Using lemma 3.1 and Theorem 3.2 we get equation (3.6).

From equation (3.2) it follows that

$$g(h(\phi X, Z), \phi W) = g(A_{\phi W} Z, \phi X) = -g(\nabla_Z W, X) = X(\ln f)g(Z, W).$$

Hence proved.

**Theorem 3.5** *The necessary and sufficient condition for a strictly proper CR-submanifold  $M$  of a indefinite  $S$ -manifold  $\tilde{M}^{2n+s}$ , tangent to the structure vector field  $\xi_\alpha$  to be locally a contact CR-warped product is that*

$$(3.7) \quad A_{\phi Z} X = (-(\phi X)(\mu) - \eta^\alpha(X))Z + \eta^\alpha(X)\eta^\alpha(Z)\xi_\alpha$$

for some function  $\mu$  on  $M$  satisfying  $W\mu = 0$  for all  $W \in D^\perp$ .

**Proof.** Let  $M = N^T \times_f N^\perp$  be a locally contact CR-warped product.

Consider  $X, Y \in D$ ,  $Z \in D^\perp$ . We can easily get  $g(A_{\phi Z} X, Y) = 0$ , which shows that  $A_{\phi Z} X$  belongs to  $D^\perp$ .

Now take any  $W \in D^\perp$ , we get

$$g(A_{\phi Z} X, W) = (-(\phi X)(\mu) - \eta^\alpha(X))g(W, Z) + \eta^\alpha(X)\eta^\alpha(Z)\eta^\alpha(W)$$

hence the result where  $\mu = \ln f$ .

Conversely, Let

$$A_{\phi Z} X = (-(\phi X)(\mu) - \eta^\alpha(X))Z + \eta^\alpha(X)\eta^\alpha(Z)\xi_\alpha.$$

We get easily that  $\tilde{g}(h(\phi X, Y), \phi Z) = 0$ ,

Also

$$\tilde{g}(h(X, W), \phi Z) = (-(\phi X)(\mu) - \eta^\alpha(X))Z + \eta^\alpha(X)\eta^\alpha(Z)\xi_\alpha,$$

where  $X, Y \in D$  and  $Z, W \in D^\perp$ . In the above equation replacing  $X$  by  $\phi X$ , we obtain

$$\tilde{g}(h(\phi X, W), \phi Z) = (X(\mu) - \eta^\alpha(X)\xi_\alpha(\mu))g(Z, W).$$

So, if we take  $X \in H(M)$  it becomes  $\tilde{g}(h(\phi X, W), \phi Z) = (X(\mu)g(Z, W))$  and if  $X = \xi_\alpha$  we get  $\tilde{g}(h(\phi X, W), \phi Z) = (1 - \delta_\alpha^\alpha)\xi_\alpha(\mu)g(Z, W)$ .

## 4 A geometric inequality for contact $CR$ -warped product in indefinite $S$ -space form

In this section, we will prove the main theorems of the paper.

Let  $M$  be a (pseudo-)Riemannian  $k$ -manifold with inner product  $\langle, \rangle$  and  $e_1, \dots, e_k$  be an orthonormal frame fields on  $M$ . For differentiable function  $\phi$  on  $M$ , the gradient  $\nabla\phi$  and the Laplacian  $\Delta\phi$  of  $\phi$  are defined respectively by

$$(4.1) \quad \langle \nabla\phi, X \rangle = X(\phi),$$

$$(4.2) \quad \Delta\phi = \sum_{j=1}^k \{(\nabla_{e_j} e_j)\phi - e_j e_j(\phi)\} = -\operatorname{div} \nabla\phi$$

for vector field  $X$  tangent to  $M$ , where  $\nabla$  is the Riemannian connection on  $M$ . As consequence, we have

$$(4.3) \quad \|\nabla\phi\|^2 = \sum_{j=1}^k (e_j(\phi))^2.$$

**Theorem 4.1** *Let  $M = N^T \times_f N^\perp$  be a contact  $CR$ -warped product of a indefinite  $S$ -space form  $\tilde{M}^{2n+s}(c)$ . Then the second fundamental form of  $M$  satisfies the following inequality*

$$(4.4) \quad \|h\|^2 \geq p\{3\|\nabla \ln f\|^2 - \Delta \ln f + (c+2)k+1\}.$$

**Proof.** We have

$$(4.6) \quad \|h(D, D^\perp)\|^2 = \sum_{j=1}^k \sum_{i=1}^p \|h(X_j, Z_i)\|^2,$$

where  $X_j$  for  $\{j = 1, \dots, k\}$  and  $Z_\alpha$  for  $\alpha = \{1, \dots, p\}$  are orthonormal frames on  $N^T$  and  $N^\perp$ , respectively. On  $N^T$  we will consider a  $\phi$ -adapted orthonormal frame, namely  $\{e_j, \phi e_j, \xi_\alpha\}$ , where  $\{j = 1, \dots, k\}, \{\alpha = 1, \dots, s\}$ .

We have to evaluate  $\|h(X, Z)\|^2$  with  $X \in D$  and  $Z \in D^\perp$ . The second fundamental form  $h(X, Z)$  is normal to  $M$  so, it splits into two orthogonal components

$$(4.7) \quad h(X, Z) = h_{\phi D^\perp}(X, Z) + h_\nu(X, Z),$$

where  $h_{\phi D^\perp}(X, Z) \in \phi D^\perp$  and  $h_\nu(X, Z) \in \nu$ . So

$$(4.8) \quad \|h(X, Z)\|^2 = \|h_{\phi D^\perp}(X, Z)\|^2 + \|h_\nu(X, Z)\|^2.$$



If  $X = \xi_\alpha$ , we have  $h(\xi_\alpha, Z) = -\phi Z$ . Hence

$$(4.9) \quad h_{\phi D^\perp}(\xi_\alpha, Z) = -\phi Z, \quad h_\nu(\xi_\alpha, Z) = 0.$$

Consider now  $X \in H(M)$  and let's compute the norm of the  $\phi D^\perp$ -component of  $h(X, Z)$ . We have

$$\|h_{\phi D^\perp}(X, Z)\|^2 = \langle h_{\phi D^\perp}(X, Z), h(X, Z) \rangle.$$

By using relation (3.7), after the computations, we obtain

$$\|h_{\phi D^\perp}(X, Z)\|^2 = (\phi X(\ln f))^2 \|Z\|^2.$$

So

$$(4.10) \quad \|h_{\phi D^\perp}(e_j, Z_i)\|^2 = (\phi e_j(\ln f))^2, \quad \|h_{\phi D^\perp}(\phi e_j, Z_i)\|^2 = (e_j(\ln f))^2.$$

On the other hand, from (4.2) we have

$$(4.11) \quad \|\nabla \ln f\|^2 = \sum_{j=1}^k (e_j(\ln f))^2 + \sum_{j=1}^k (\phi e_j(\ln f))^2.$$

Since  $\xi_\alpha(\ln f) = 0$ . Finally we can compute the norm  $\|h_{\phi D^\perp}(D, D^\perp)\|^2$ . Thus

$$\begin{aligned} \|h_{\phi D^\perp}(D, D^\perp)\|^2 &= \sum_{j=1}^k \sum_{i=1}^p \{ \|h_{\phi D^\perp}(e_j, Z_i)\|^2 + \|h_{\phi D^\perp}(\phi e_j, Z_i)\|^2 \} + \sum_{\alpha=1}^s \sum_{i=1}^p \|h_{\phi D^\perp}(\xi_\alpha, Z_i)\|^2 = \\ &= \sum_{i=1}^p \|\nabla \ln f\|^2 + \sum_{i=1}^p \|\phi Z_i\|^2. \end{aligned}$$

Since  $\|\phi Z_i\|^2 = 1$  we can conclude that

$$(4.12) \quad \|h_{\phi D^\perp}(D, D^\perp)\|^2 = \left\{ \sum_{i=1}^p \|\nabla \ln f\|^2 + 1 \right\} p.$$

Now we will compute the norm of the  $\nu$ -component of  $h(X, Z)$ . We have

$$\|h_\nu(X, Z)\|^2 = \langle h_\nu(X, Z), h(X, Z) \rangle = \langle A_{h_\nu(X, Z)} X, Z \rangle,$$

by using lemma (3.3) we can write  $A_{h_\nu(X, Z)} X = A_{\phi h_\nu(X, Z)}(\phi X)$  so,

$$\|h_\nu(X, Z)\|^2 = \langle \phi h(X, Z) - \phi h_{\phi D^\perp}(X, Z), h(\phi X, Z) \rangle.$$

Since  $\phi h_{\phi D^\perp}(X, Z)$  belongs to  $D^\perp$  we obtain

$$(4.13) \quad \|h_\nu(X, Z)\|^2 = \tilde{g}(\phi h(X, Z), h(\phi X, Z)), \quad \forall X \in H(M), \quad Z \in D^\perp.$$

Consider the tensor field  $\tilde{H}_B$ . As we already have seen

$$\tilde{H}_B(X, Z) = \langle (\nabla_{\phi X})h(X, Z) - (\nabla_X h)(\phi X, Z), \phi Z \rangle, \quad \forall X \in H(M), \quad Z \in D^\perp.$$

Using the definition of  $\nabla h$ , developing the expression above we obtain

$$\begin{aligned}\tilde{H}_B(X, Z) = & \langle \nabla^\perp_{\phi X} h(X, Z), \phi Z \rangle - \langle h(\nabla_{\phi X} X, Z), \phi Z \rangle - \langle h(X, \nabla_{\phi X} Z), \phi Z \rangle \\ & - \langle \nabla^\perp_X h(\phi X, Z), \phi Z \rangle + \langle h(\nabla_X \phi X, Z), \phi Z \rangle + \langle h(\phi X, \nabla_X Z), \phi Z \rangle,\end{aligned}$$

after using lemma 3.4 and theorem 3.5, we get

$$\begin{aligned}\tilde{H}_B(X, Z) = & \|Z\|^2 \{(\phi X(\ln f))^2 - (\phi X)(\phi X(\ln f)) + (X \ln f)^2 - X(X \ln f) \\ & + (\phi \nabla_{\phi X} X)(\ln f) - \|X\|^2 - (\phi X \nabla_X \phi X)(\ln f) - \|X\|^2 \\ & + (\phi X(\ln f))^2 + (X(\ln f))^2\} + 2 \langle \phi h(X, Z), h(\phi X, Z) \rangle,\end{aligned}$$

which becomes

$$\begin{aligned}\tilde{H}_B(X, Z) = & \|Z\|^2 \{2(\phi X(\ln f))^2 - (\phi X)^2(\ln f) + 2(X \ln f)^2 - X^2(\ln f) \\ (4.14) \quad & + ((\phi \nabla_{\phi X} X) - (\phi X \nabla_X \phi X))(\ln f) - 2\|X\|^2\} + 2\|h_\nu(X, Z)\|^2.\end{aligned}$$

We can easily prove that

$$(4.15) \quad (\phi \nabla_{\phi X} X)(\ln f) = (\nabla_{\phi X} \phi X)(\ln f), \quad (\phi \nabla_X \phi X)(\ln f) = -\nabla_X X(\ln f).$$

Using (4.14) and (4.15), we get

$$\begin{aligned}(4.16) \quad \tilde{H}_B(X, Z) = & \|Z\|^2 \{2(\phi X(\ln f))^2 - (\phi X)^2(\ln f) + 2(X \ln f)^2 - X^2(\ln f) \\ & + (\nabla_{\phi X}(\phi X) + \nabla_X X)(\ln f) - 2\|X\|^2\} + 2\|h_\nu(X, Z)\|^2.\end{aligned}$$

Using orthonormal frames, we have

$$\begin{aligned}(4.17) \quad \tilde{H}_B(e_j, Z_i) = & \|Z_i\|^2 \{2(\phi e_j(\ln f))^2 - (\phi e_j)^2(\ln f) + 2(e_j \ln f)^2 - e_j^2(\ln f) \\ & + (\nabla_{\phi e_j}(\phi e_j) + \nabla_{e_j} e_j)(\ln f) - 2\|X_j\|^2\} + 2\|h_\nu(e_j, Z_i)\|^2.\end{aligned}$$

Similarly,

$$\begin{aligned}(4.18) \quad \tilde{H}_B(\phi e_j, Z_i) = & \|Z_i\|^2 \{2(e_j(\ln f))^2 - e_j^2(\ln f) + 2(\phi e_j \ln f)^2 - (\phi e_j)^2(\ln f) \\ & + (\nabla_{e_j} e_j + \nabla_{\phi e_j}(\phi e_j))(\ln f) - 2\|X_j\|^2\} + 2\|h_\nu(\phi e_j, Z_i)\|^2.\end{aligned}$$

On the other hand we have

$$\triangle(\ln f) = \sum_{j=1}^k \{(\nabla_{e_j} e_j)(\ln f) - e_j^2(\ln f)\} + \sum_{j=1}^k \{(\nabla_{\phi e_j} \phi e_j)(\ln f) - \phi e_j^2(\ln f)\}$$

by using (4.3), we get

$$2\|\nabla \ln f\|^2 = 2 \sum_{j=1}^k (e_j(\ln f))^2 + 2 \sum_{j=1}^k (\phi e_j(\ln f))^2.$$

Since  $\xi_\alpha \ln f = 0$ . Taking the sum of (4.17) and (4.18) we get

$$2 \sum_{j=1}^k \sum_{i=1}^p \{ \|h_\nu(e_j, Z_i)\|^2 + \|h_\nu(\phi e_j, Z_i)\|^2 \} = \sum_{j=1}^k \sum_{i=1}^p \{ \tilde{H}_B(e_j, Z_i) + \tilde{H}_B(\phi e_j, Z_i) \} \\ - 2p\Delta(\ln f) + 4kp + 4\|\nabla \ln f\|^2,$$

by using proposition 2.11, we have

$$(4.19) \quad \sum_{j=1}^k \sum_{i=1}^p \{ \|h_\nu(e_j, Z_i)\|^2 + \|h_\nu(\phi e_j, Z_i)\|^2 \} = ckp - p\Delta(\ln f) + 2kp + 2k\|\nabla \ln f\|^2.$$

Now from (4.8), (4.12) and (4.19) we conclude that  $h$  satisfies the inequality.

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